

# Discrete Torsion and Branes in M-theory from Mathematical Viewpoint

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## **abstract**

We study orbifold group actions on locally defined fields upon M-theory branes in a three-form C-fields background. We derive some constraints from the consistency of the orbifold group actions. We show the possibility of the existence of M-theory discrete torsion for the fields on the worldvolume and discuss its features.

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## 1. Introduction

NS-NS B-fields are studied in various contexts, particularly noncommutative geometry and discrete torsion. The relation between the noncommutative geometry and the B-fields has been mentioned in [1] recently. When non-trivial B-fields exist along a D-brane, a gauge theory on the D-brane has noncommutativity. On the other hand discrete torsion was originally pointed out in [2] and the orbifold group action on a worldvolume gauge theory on D-branes at the orbifold singularity has been studied recently in [3,4]. In this direction a lot of works [5–9] have been done, and this paper also has a great interest in this subject.

Since orbifold theories have twisted sectors, constraints for modular invariance on a one-loop partition function of a closed string on an orbifold  $M/\Gamma$ , which is a quotient space of a manifold  $M$  by an orbifold group  $\Gamma$ , induce a degree of freedom, which is called discrete torsion  $\epsilon(g, h) \in U(1)$  for  $g, h \in \Gamma$  [2]. Higher loop modular invariance requires the following constraints

$$\begin{aligned}\epsilon(g_1 g_2, g_3) &= \epsilon(g_1, g_3) \epsilon(g_2, g_3), \\ \epsilon(g, h) &= \epsilon(h, g)^{-1}, \\ \epsilon(g, g) &= 1,\end{aligned}\tag{1.1}$$

where  $\epsilon(g, h)$  is classified by a second group cohomology  $H^2(\Gamma, U(1))$ .

Discrete torsion for open strings has been shown in [3,4]. For the orbifold  $M/\Gamma$ , supersymmetric Yang-Mills fields  $\phi$  in the worldvolume gauge theory on D-branes at the orbifold singularities are projected by  $\Gamma$  as

$$\gamma(g)^{-1} \phi \gamma(g) = r(g) \phi, \quad g \in \Gamma,$$

where  $\gamma(g)$  is a representation of  $\Gamma$  in the gauge group and  $r(g)$  is a space-time action of  $\Gamma$ . Then  $\gamma(g)$  is in a projective representation

$$\gamma(g) \gamma(h) = \epsilon(g, h) \gamma(gh).\tag{1.2}$$

The moduli space of the gauge theory has the same structure as the expectations based on [10]. More studies on non-abelian orbifolds with discrete torsion have been done in [5,6] in terms of Schur Multipliers.

String theories have physical and mathematical aspects. Mathematical understanding of discrete torsion has been proposed in [11–16], which define the B-fields on each local

patch in terms of gerbes and state that discrete torsion is the choice of orbifold group action on the B-fields. Since there exist gauge symmetries, it is in fact not sufficient to define the orbifold action only on the base space. It is necessary to choose the action on fields as well. These remarks include more applications. For example the choice of orbifold group action on vector fields gives rise to degrees of freedom which are known as orbifold Wilson lines.

In Section 2 we briefly review these mathematical aspects of discrete torsion and branes in the string theories. Section 3 is devoted for the calculations of orbifold group actions on fields in M-theory. In Section 4 we present summaries and conclusions with some discussions on problems left for further study.

## 2. Review of discrete torsion and branes in string theories

In this section we review the mathematical aspects of discrete torsion and branes proposed in [11–13]. When we define the NS-NS B-fields, we need a two-form field  $B^\alpha$  on a patch  $U_\alpha$ , a one-form  $p^{\alpha\beta}$  on  $U_{\alpha\beta}(\equiv U_\alpha \cap U_\beta)$  and a  $U(1)$ -valued function  $q_{\alpha\beta\gamma}$  on  $U_{\alpha\beta\gamma}(\equiv U_\alpha \cap U_\beta \cap U_\gamma)$  which satisfy the following equations

$$B^\alpha - B^\beta = dp^{\alpha\beta}, \quad (2.1)$$

$$p^{\alpha\beta} + p^{\beta\gamma} + p^{\gamma\alpha} = d \log q_{\alpha\beta\gamma}, \quad (2.2)$$

$$\delta(q_{\alpha\beta\gamma}) = 1. \quad (2.3)$$

From Eq.(2.3), in order to preserve Čech cocycle  $q_{\alpha\beta\gamma}$  by an orbifold group action  $g \in \Gamma$ , up to coboundaries, we require that the pullback of  $q_{\alpha\beta\gamma}$  becomes

$$g^* q_{\alpha\beta\gamma} = q_{\alpha\beta\gamma} \nu_{\alpha\beta}^g \nu_{\beta\gamma}^g \nu_{\gamma\alpha}^g, \quad (2.4)$$

where  $\nu_{\alpha\beta}^g$  are some Čech cochains for each  $g$ . From Eqs.(2.2) and (2.4), we obtain the orbifold group action on  $p^{\alpha\beta}$  as

$$g^* p^{\alpha\beta} = p^{\alpha\beta} + d \log \nu_{\alpha\beta}^g + \Lambda(g)^\alpha - \Lambda(g)^\beta, \quad (2.5)$$

for some one-forms  $\Lambda(g)^\alpha$ . Using Eqs.(2.1) and (2.5), the orbifold group action on  $B^\alpha$  becomes

$$g^* B^\alpha = B^\alpha + d\Lambda(g)^\alpha. \quad (2.6)$$

Expanding  $(g_1 g_2)^* q_{\alpha\beta\gamma}$ ,  $\nu_{\alpha\beta}^{g_1 g_2 g_3}$ ,  $(g_1 g_2)^* p^{\alpha\beta}$  and  $(g_1 g_2)^* B^\alpha$  respectively in two different ways, we obtain some constraints

$$\begin{aligned}\Lambda(g_1 g_2)^\alpha &= \Lambda(g_2)^\alpha + g_2^* \Lambda(g_1)^\alpha - d \log h_\alpha^{g_1, g_2}, \\ \nu_{\alpha\beta}^{g_1 g_2} &= \nu_{\alpha\beta}^{g_2} \left( g_2^* \nu_{\alpha\beta}^{g_1} \right) (h_\alpha^{g_1, g_2}) \left( h_\beta^{g_1, g_2} \right)^{-1}, \\ (h_\alpha^{g_1, g_2 g_3}) (h_\alpha^{g_2, g_3}) &= (g_3^* h_\alpha^{g_1, g_2}) (h_\alpha^{g_1 g_2, g_3}),\end{aligned}$$

where  $h_\alpha^{g_1, g_2}$  are some Čech cochains.

Orbifold Wilson surfaces  $\exp(\int B)$  which appear in the one-loop partition function, are the analogues of orbifold Wilson loops and give rise to phases. Now let us take two kinds of definitions of the orbifold group actions. We describe  $h_\alpha^{g, h}$  in one definition and  $\bar{h}_\alpha^{g, h}$  in the other. In order to consider the difference between these two definitions we use

$$\omega^{g, h} = \frac{h^{g, h}}{\bar{h}^{g, h}}.$$

Then the phases lead to

$$(\omega^{g, h}) (\omega^{h, g})^{-1}. \quad (2.7)$$

Note that we assume the B-fields are completely trivial so that  $h_\alpha^{g, h}$  and  $\omega_\alpha^{g, h}$  are globally defined. Eq.(2.7) stands for the phases from the contribution of twisted sectors to the partition function, in other words, it corresponds to discrete torsion introduced in [2]. In fact the phases (2.7) satisfy the conditions (1.1) for the modular invariance.

Next we consider the orbifold group action on  $N$  coincident D-branes. There are  $U(N)$  gauge fields, which come from the Chan-Paton factors of open strings ending on the D-branes. Since gauge transformations associate the gauge fields  $A$  with the B-fields, the gauge fields also should be defined on local patches. From [17] we can link  $A^\alpha$  to  $B^\alpha$  by using the following equations

$$A^\alpha - g_{\alpha\beta} A^\beta g_{\alpha\beta}^{-1} - d \log g_{\alpha\beta}^{-1} = p^{\alpha\beta} I, \quad (2.8)$$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = q_{\alpha\beta\gamma} I, \quad (2.9)$$

where  $g_{\alpha\beta}$  is a  $N \times N$  matrix and  $I$  is a unit matrix.  $g_{\alpha\beta}$  is a transition function for the gauge bundle  $A$  when the B-fields are completely trivial.

From Eqs.(2.4),(2.5),(2.8) and (2.9) the orbifold actions on the gauge fields become

$$g^* A^\alpha = (\gamma_\alpha^g) A^\alpha (\gamma_\alpha^g)^{-1} + (\gamma_\alpha^g) d(\gamma_\alpha^g)^{-1} + I \Lambda(g)^\alpha, \quad (2.10)$$

$$g^* g_{\alpha\beta} = \left( \nu_{\alpha\beta}^g \right) \left[ (\gamma_\alpha^g) g_{\alpha\beta} (\gamma_\beta^g)^{-1} \right], \quad (2.11)$$

for some  $N \times N$  matrices  $\gamma_\alpha^g$ . Expanding  $(g_1 g_2)^* g_{\alpha\beta}$  in two different ways, we obtain a constraint

$$(g_2^* \gamma_\alpha^{g_1})(\gamma_\alpha^{g_2}) = h_\alpha^{g_1, g_2} (\gamma_\alpha^{g_1 g_2}). \quad (2.12)$$

Let us consider the completely trivial B-fields given by  $B^\alpha = 0$ ,  $p^{\alpha\beta} = 0$  and  $q_{\alpha\beta\gamma} = 1$ . Then it is meaningful that we set the bundle on the D-branes to be topologically trivial. We can replace the locally defined gauge field  $A^\alpha$  with a globally defined  $U(N)$  gauge field  $A$  and set  $g_{\alpha\beta} = 1$ . If furthermore the gauge field  $A$  is constant, we can assume that  $\gamma^g$  and  $h^{g_1, g_2}$  are constant. From Eq.(2.12) we obtain

$$(\gamma^{g_1})(\gamma^{g_2}) = h^{g_1, g_2} (\gamma^{g_1 g_2}).$$

$\gamma^g$  has a projective representation. And  $h^{g_1, g_2}$  is classified by  $H^2(\Gamma, U(1))$ . These results are in good agreement with [3,4].

### 3. Discrete torsion and branes in M-theory

In M-theory there exist membranes and M5-branes, and the membranes are considered as three-form C-fields in an eleven dimensional supergravity. Since the C-fields compactified on  $S^1$  lead to the B-fields in the string theories, membrane twisted sector would derive discrete torsion in M-theory. In the string theories discrete torsion, which is classified by  $H^2(\Gamma, U(1))$ , appears as the phases derived from the term  $\exp(\int B)$ , while in M-theory the term  $\exp(\int C)$  leads to some phases. In [14] the phases have been calculated from the contribution of membrane twisted sectors on  $T^3$  in terms of 2-gerbes and have been classified by a third group cohomology  $H^3(\Gamma, U(1))$ .

In order to define the C-fields on each patch, we need a three-form C-field  $C^\alpha$  on a patch  $U_\alpha$ , a two-form  $u^{\alpha\beta}$  on  $U_{\alpha\beta}$ , a one-form  $v^{\alpha\beta\gamma}$  on  $U_{\alpha\beta\gamma}$  and a function  $h_{\alpha\beta\gamma\delta}$  on  $U_{\alpha\beta\gamma\delta}(\equiv U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta)$ . The forms and the function are related by the following equations

$$C^\alpha - C^\beta = du^{\alpha\beta}, \quad (3.1)$$

$$u^{\alpha\beta} + u^{\beta\gamma} + u^{\gamma\alpha} = dv^{\alpha\beta\gamma}, \quad (3.2)$$

$$v^{\beta\gamma\delta} - v^{\alpha\gamma\delta} + v^{\alpha\beta\delta} - v^{\alpha\beta\gamma} = d \log h_{\alpha\beta\gamma\delta}, \quad (3.3)$$

$$\delta h_{\alpha\beta\gamma\delta} = 1. \quad (3.4)$$

The actions of  $g \in \Gamma$  for the C-fields described in [14] are

$$g^* C^\alpha = C^\alpha + d\Lambda^{(2)}(g)^\alpha, \quad (3.5)$$

$$g^* u^{\alpha\beta} = u^{\alpha\beta} + d\Lambda^{(1)}(g)^{\alpha\beta} + \Lambda^{(2)}(g)^\alpha - \Lambda^{(2)}(g)^\beta, \quad (3.6)$$

$$g^* v^{\alpha\beta\gamma} = v^{\alpha\beta\gamma} + \Lambda^{(1)}(g)^{\alpha\beta} + \Lambda^{(1)}(g)^{\beta\gamma} + \Lambda^{(1)}(g)^{\gamma\alpha} + d \log \nu_{\alpha\beta\gamma}^g, \quad (3.7)$$

$$g^* h_{\alpha\beta\gamma\delta} = h_{\alpha\beta\gamma\delta} \left( \nu_{\beta\gamma\delta}^g \right) \left( \nu_{\alpha\gamma\delta}^g \right)^{-1} \left( \nu_{\alpha\beta\delta}^g \right) \left( \nu_{\alpha\beta\gamma}^g \right)^{-1}, \quad (3.8)$$

where, for each element  $g$  in the orbifold group  $\Gamma$ ,  $\nu_{\alpha\beta\gamma}^g$  are some Čech cochains,  $\Lambda^{(1)}(g)^{\alpha\beta}$  are some local one-forms and  $\Lambda^{(2)}(g)^\alpha$  are some local two-forms. And we obtain the constraints [14],

$$\Lambda^{(2)}(g_1 g_2)^\alpha = \Lambda^{(2)}(g_2)^\alpha + g_2^* \Lambda^{(2)}(g_1)^\alpha + d\Lambda^{(3)}(g_1, g_2)^\alpha, \quad (3.9)$$

$$\begin{aligned} \Lambda^{(1)}(g_1 g_2)^{\alpha\beta} &= \Lambda^{(1)}(g_2)^{\alpha\beta} + g_2^* \Lambda^{(1)}(g_1)^{\alpha\beta} - \Lambda^{(3)}(g_1, g_2)^\alpha \\ &\quad + \Lambda^{(3)}(g_1, g_2)^\beta - d \log \lambda_{\alpha\beta}^{g_1, g_2}, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \Lambda^{(3)}(g_2, g_3)^\alpha + \Lambda^{(3)}(g_1, g_2 g_3)^\alpha &= g_3^* \Lambda^{(3)}(g_1, g_2)^\alpha + \Lambda^{(3)}(g_1 g_2, g_3)^\alpha \\ &\quad + d \log \gamma_\alpha^{g_1, g_2, g_3}, \end{aligned} \quad (3.11)$$

$$\nu_{\alpha\beta\gamma}^{g_1 g_2} = \left( \nu_{\alpha\beta\gamma}^{g_2} \right) \left( g_2^* \nu_{\alpha\beta\gamma}^{g_1} \right) \left( \lambda_{\alpha\beta}^{g_1, g_2} \right) \left( \lambda_{\beta\gamma}^{g_1, g_2} \right) \left( \lambda_{\gamma\alpha}^{g_1, g_2} \right), \quad (3.12)$$

$$\left( \lambda_{\alpha\beta}^{g_1 g_2, g_3} \right) \left( g_3^* \lambda_{\alpha\beta}^{g_1, g_2} \right) = \left( \lambda_{\alpha\beta}^{g_1, g_2 g_3} \right) \left( \lambda_{\alpha\beta}^{g_2, g_3} \right) \left( \gamma_\alpha^{g_1, g_2, g_3} \right) \left( \gamma_\beta^{g_1, g_2, g_3} \right)^{-1}, \quad (3.13)$$

$$\left( \gamma_\alpha^{g_1, g_2, g_3 g_4} \right) \left( \gamma_\alpha^{g_1 g_2, g_3, g_4} \right) = \left( \gamma_\alpha^{g_1, g_2 g_3, g_4} \right) \left( \gamma_\alpha^{g_2, g_3, g_4} \right) \left( g_4^* \gamma_\alpha^{g_1, g_2, g_3} \right). \quad (3.14)$$

In the similar way to Section 2 the difference of orbifold group actions  $\omega_\alpha^{g_1, g_1, g_3} = \gamma_\alpha^{g_1, g_2, g_3} / \bar{\gamma}_\alpha^{g_1, g_2, g_3}$  leads to the membrane twisted sector phase on  $T^3$  and is classified by  $H^3(\Gamma, U(1))$  for the completely trivial C-fields [14].

Now let us consider the orbifold group actions on fields in a worldvolume theory on branes. When a membrane ends on the branes as a string, the end line is assumed as a two-form field  $B$  in the worldvolume theory. Since the three-form field  $C$  is mapped to  $C + dB$  by gauge transformations, on the analogy of the string theories the B-fields are also to be defined on each local patch in terms of a two-form  $B^\alpha$ , a one-form  $p^{\alpha\beta}$  and a function  $q_{\alpha\beta\gamma}$ . From [17] we are able to associate the B-fields with the C-fields as

$$B^\alpha - B^\beta + dp^{\alpha\beta} = u^{\alpha\beta}, \quad (3.15)$$

$$p^{\alpha\beta} + p^{\beta\gamma} + p^{\gamma\alpha} + d \log q_{\alpha\beta\gamma} = v^{\alpha\beta\gamma}, \quad (3.16)$$

$$(q_{\beta\gamma\delta}) (q_{\alpha\gamma\delta})^{-1} (q_{\alpha\beta\delta}) (q_{\alpha\beta\gamma})^{-1} = h_{\alpha\beta\gamma\delta}. \quad (3.17)$$

We calculate the orbifold group actions on the B-fields. Firstly we suppose the action on  $q_{\alpha\beta\gamma}$  as

$$g^* q_{\alpha\beta\gamma} = q_{\alpha\beta\gamma} \left( \mu_{\alpha\beta}^g \right) \left( \mu_{\beta\gamma}^g \right) \left( \mu_{\gamma\alpha}^g \right),$$

so that  $q_{\alpha\beta\gamma}$  is preserved up to coboundaries.  $\mu_{\alpha\beta}^g$  are some Čech cochains for each  $g$ . Then the pullback of the left hand side of Eq.(3.17) becomes

$$g^* \left[ (q_{\beta\gamma\delta}) (q_{\alpha\gamma\delta})^{-1} (q_{\alpha\beta\delta}) (q_{\alpha\beta\gamma})^{-1} \right] = (q_{\beta\gamma\delta}) (q_{\alpha\gamma\delta})^{-1} (q_{\alpha\beta\delta}) (q_{\alpha\beta\gamma})^{-1} = h_{\alpha\beta\gamma\delta},$$

while from Eq.(3.8) the pullback of the right hand side of Eq.(3.17) has additional factors  $\nu^g$ . So we should instead define the orbifold action on  $q_{\alpha\beta\gamma}$  as

$$g^* q_{\alpha\beta\gamma} = \left( \nu_{\alpha\beta\gamma}^g \right) q_{\alpha\beta\gamma} \left( \mu_{\alpha\beta}^g \right) \left( \mu_{\beta\gamma}^g \right) \left( \mu_{\gamma\alpha}^g \right). \quad (3.18)$$

From Eqs.(3.7), (3.16) and (3.18) we obtain

$$g^* p^{\alpha\beta} = p^{\alpha\beta} - d \log \mu_{\alpha\beta}^g + \Lambda^{(1)}(g)^{\alpha\beta} + \Lambda^{(1)}(g)^\alpha - \Lambda^{(1)}(g)^\beta, \quad (3.19)$$

for some local one-forms  $\Lambda^{(1)}(g)^\alpha$ . Using Eqs.(3.6), (3.15) and (3.19), we calculate the orbifold group action on  $B^\alpha$  as

$$g^* B^\alpha = B^\alpha - d\Lambda^{(1)}(g)^\alpha + \Lambda^{(2)}(g)^\alpha. \quad (3.20)$$

$\mu_{\alpha\beta}^g$  and  $\Lambda^{(1)}(g)^\alpha$  determine the structure of orbifold group action on the B-fields.

If a membrane extends in a two dimensional subspace transverse to the branes and ends on the branes as a point, we can consider the end point as a one-form field  $A$  in the worldvolume gauge theory on the branes. From [17] we write down the relations between the B-fields and the A-fields,

$$A^\alpha - g_{\alpha\beta} A^\beta g_{\alpha\beta}^{-1} - d \log g_{\alpha\beta} = p^{\alpha\beta}, \quad (3.21)$$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = q_{\alpha\beta\gamma}. \quad (3.22)$$

$A^\alpha$  and  $g_{\alpha\beta}$  are the data of A-fields described on local patches in the same way as Section 2. We suppose the orbifold group action on  $g_{\alpha\beta}$  as

$$g^* g_{\alpha\beta} = (\rho_\alpha^g) g_{\alpha\beta} \left( \rho_\beta^g \right)^{-1}, \quad (3.23)$$

where  $\rho^g$  are some functions. Since from the left hand side of Eq.(3.22) we obtain

$$g^* [g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha}] = g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = q_{\alpha\beta\gamma},$$

we need additional factors in Eq.(3.23) in order for the above equation to be consistent with Eq.(3.18). When  $\nu_{\alpha\beta\gamma}^g$  is equal to one, we are able to define the orbifold action on  $g_{\alpha\beta}$  as

$$g^* g_{\alpha\beta} = \left( \mu_{\alpha\beta}^g \right) (\rho_\alpha^g) g_{\alpha\beta} \left( \rho_\beta^g \right)^{-1}. \quad (3.24)$$

If  $\Lambda^{(1)}(g)^{\alpha\beta}$  vanishes, we can obtain the action of  $g$  on  $A^\alpha$  as

$$g^* A^\alpha = (\rho_\alpha^g) A^\alpha (\rho_\alpha^g)^{-1} + d \log \rho_\alpha^g + \Lambda^{(1)}(g)^\alpha. \quad (3.25)$$

These two requirements,  $\nu_{\alpha\beta\gamma}^g = 1$  and  $\Lambda^{(1)}(g)^{\alpha\beta} = 0$ , are realized, for example, when the C-fields are topologically trivial.

Next we derive some constraints. We calculate  $(g_1 g_2)^* q_{\alpha\beta\gamma}$  for  $g_1, g_2 \in \Gamma$  in two different ways and they become

$$\begin{aligned} g_2^* (g_1^* q_{\alpha\beta\gamma}) &= \left( \nu_{\alpha\beta\gamma}^{g_2} \right) q_{\alpha\beta\gamma} \left( g_2^* \nu_{\alpha\beta\gamma}^{g_1} \right) \\ &\quad \times \left( \mu_{\alpha\beta}^{g_2} \right) \left( g_2^* \mu_{\alpha\beta}^{g_1} \right) \left( \mu_{\beta\gamma}^{g_2} \right) \left( g_2^* \mu_{\beta\gamma}^{g_1} \right) \left( \mu_{\gamma\alpha}^{g_2} \right) \left( g_2^* \mu_{\gamma\alpha}^{g_1} \right), \\ (g_1 g_2)^* q_{\alpha\beta\gamma} &= \left( \nu_{\alpha\beta\gamma}^{g_2} \right) q_{\alpha\beta\gamma} \left( g_2^* \nu_{\alpha\beta\gamma}^{g_1} \right) \\ &\quad \times \left( \lambda_{\alpha\beta}^{g_1, g_2} \right) \left( \lambda_{\beta\gamma}^{g_1, g_2} \right) \left( \lambda_{\gamma\alpha}^{g_1, g_2} \right) \left( \mu_{\alpha\beta}^{g_1 g_2} \right) \left( \mu_{\beta\gamma}^{g_1 g_2} \right) \left( \mu_{\gamma\alpha}^{g_1 g_2} \right). \end{aligned}$$

Comparing these equations, we read the following constraint

$$\left( \mu_{\alpha\beta}^{g_2} \right) \left( g_2^* \mu_{\alpha\beta}^{g_1} \right) = \left( \lambda_{\alpha\beta}^{g_1, g_2} \right) \left( \mu_{\alpha\beta}^{g_1 g_2} \right) (\theta_\alpha^{g_1, g_2})^{-1} \left( \theta_\beta^{g_1, g_2} \right), \quad (3.26)$$

where  $\theta_\alpha^{g_1, g_2}$  are some functions. We also compute the pullbacks of  $p^{\alpha\beta}$  by  $g_1 g_2$ ,

$$\begin{aligned} g_2^* (g_1^* p^{\alpha\beta}) &= p^{\alpha\beta} - d \log \mu_{\alpha\beta}^{g_2} + \Lambda^{(1)}(g_2)^{\alpha\beta} + \Lambda^{(1)}(g_2)^\alpha - \Lambda^{(1)}(g_2)^\beta \\ &\quad - g_2^* \left( d \log \mu_{\alpha\beta}^{g_1} \right) + g_2^* \Lambda^{(1)}(g_1)^{\alpha\beta} + g_2^* \Lambda^{(1)}(g_1)^\alpha - g_2^* \Lambda^{(1)}(g_1)^\beta, \\ (g_1 g_2)^* p^{\alpha\beta} &= p^{\alpha\beta} - d \log \mu_{\alpha\beta}^{g_1 g_2} + \Lambda^{(1)}(g_1 g_2)^{\alpha\beta} + \Lambda^{(1)}(g_1 g_2)^\alpha - \Lambda^{(1)}(g_1 g_2)^\beta \\ &= p^{\alpha\beta} - d \log \theta_\alpha^{g_1, g_2} + d \log \theta_\beta^{g_1, g_2} - d \log \mu_{\alpha\beta}^{g_2} - g_2^* \left( d \log \mu_{\alpha\beta}^{g_1} \right) \\ &\quad + \Lambda^{(1)}(g_2)^{\alpha\beta} + g_2^* \Lambda^{(1)}(g_1)^{\alpha\beta} - \Lambda^{(3)}(g_1, g_2)^\alpha + \Lambda^{(3)}(g_1, g_2)^\beta \\ &\quad + \Lambda^{(1)}(g_1 g_2)^\alpha - \Lambda^{(1)}(g_1 g_2)^\beta. \end{aligned}$$

Since  $g_2^*(g_1^*p^{\alpha\beta})$  is equal to  $(g_1g_2)^*p^{\alpha\beta}$ , we obtain the constraint

$$\Lambda^{(1)}(g_2)^\alpha + g_2^*\Lambda^{(1)}(g_1)^\alpha = \Lambda^{(1)}(g_1g_2)^\alpha - \Lambda^{(3)}(g_1, g_2)^\alpha - d \log \theta_\alpha^{g_1, g_2} \quad (3.27)$$

For  $g_{\alpha\beta}$  we calculate the following pullbacks as

$$\begin{aligned} g_2^*(g_1^*g_{\alpha\beta}) &= \left(g_2^*\mu_{\alpha\beta}^{g_1}\right) \left(\mu_{\alpha\beta}^{g_2}\right) (g_2^*\rho_\alpha^{g_1}) (\rho_\alpha^{g_2}) g_{\alpha\beta} \left(\rho_\beta^{g_2}\right)^{-1} \left(g_2^*\rho_\beta^{g_1}\right)^{-1} \\ &= \left(\lambda_{\alpha\beta}^{g_1, g_2}\right) \left(\mu_{\alpha\beta}^{g_1 g_2}\right) (\theta_\alpha^{g_1, g_2})^{-1} \left(\theta_\beta^{g_1, g_2}\right) \\ &\quad \times (g_2^*\rho_\alpha^{g_1}) (\rho_\alpha^{g_2}) g_{\alpha\beta} \left(\rho_\beta^{g_2}\right)^{-1} \left(g_2^*\rho_\beta^{g_1}\right)^{-1}, \\ (g_1g_2)^*g_{\alpha\beta} &= \left(\mu_{\alpha\beta}^{g_1 g_2}\right) (\rho_\alpha^{g_1 g_2}) g_{\alpha\beta} \left(\rho_\beta^{g_1 g_2}\right)^{-1}. \end{aligned}$$

From these equations, when  $\lambda_{\alpha\beta}^{g_1, g_2}$  becomes one, we can obtain

$$(g_2^*\rho_\alpha^{g_1}) (\rho_\alpha^{g_2}) = (\theta_\alpha^{g_1, g_2}) (\rho_\alpha^{g_1 g_2}). \quad (3.28)$$

We compare  $\mu_{\alpha\beta}^{(g_1g_2)g_3}$  with  $\mu_{\alpha\beta}^{g_1(g_2g_3)}$ . These two terms become

$$\begin{aligned} \mu_{\alpha\beta}^{(g_1g_2)g_3} &= \left(\mu_{\alpha\beta}^{g_3}\right) \left(g_3^*\mu_{\alpha\beta}^{g_2}\right) \left((g_2g_3)^*\mu_{\alpha\beta}^{g_1}\right) \left(\lambda_{\alpha\beta}^{g_1 g_2, g_3}\right)^{-1} \left(g_3^*\lambda_{\alpha\beta}^{g_1, g_2}\right)^{-1} \\ &\quad \times (\theta_\alpha^{g_1 g_2, g_3}) (g_3^*\theta_\alpha^{g_1, g_2}) \left(\theta_\beta^{g_1 g_2, g_3}\right)^{-1} \left(g_3^*\theta_\beta^{g_1, g_2}\right)^{-1}, \\ \mu_{\alpha\beta}^{g_1(g_2g_3)} &= \left(\mu_{\alpha\beta}^{g_3}\right) \left(g_3^*\mu_{\alpha\beta}^{g_2}\right) \left((g_2g_3)^*\mu_{\alpha\beta}^{g_1}\right) \left(\lambda_{\alpha\beta}^{g_1, g_2 g_3}\right)^{-1} \left(\lambda_{\alpha\beta}^{g_2, g_3}\right)^{-1} \\ &\quad \times (\theta_\alpha^{g_1, g_2 g_3}) (\theta_\alpha^{g_2, g_3}) \left(\theta_\beta^{g_1, g_2 g_3}\right)^{-1} \left(\theta_\beta^{g_2, g_3}\right)^{-1}. \end{aligned}$$

From these equations and Eq.(3.13) we obtain the constraint

$$(\theta_\alpha^{g_1 g_2, g_3}) (g_3^*\theta_\alpha^{g_1, g_2}) = \gamma_\alpha^{g_1, g_2, g_3} (\theta_\alpha^{g_1, g_2 g_3}) (\theta_\alpha^{g_2, g_3}). \quad (3.29)$$

Note that we have required  $\nu_{\alpha\beta\gamma}^g = 1$  and  $\lambda_{\alpha\beta}^{g_1, g_2} = 1$  and these conditions are consistent with the constraint (3.12).

#### 4. Conclusions and discussion

A lot of works on orbifolds and discrete torsion have been done in the string theories, but we do not know precisely these subjects in M-theory. So we considered orbifold and discrete torsion in M-theory on the analogy of the string theories.

We used the results shown in [14], where the three-form C-fields with connections were presented in terms of the three-forms  $C^\alpha$ , the two-forms  $u^{\alpha\beta}$ , the one-forms  $v^{\alpha\beta\gamma}$  and the functions  $h_{\alpha\beta\gamma\delta}$  defined on local patches. In [14] the two-forms  $\Lambda^{(2)}(g)^\alpha$ , the one-forms  $\Lambda^{(1)}(g)^{\alpha\beta}$  and the Čech cochains  $\nu_{\alpha\beta\gamma}^g$  were also introduced as the structures describing the actions of orbifold group  $\Gamma$  on the C-fields. For the constraints we introduced  $\Lambda^{(3)}(g_1, g_2)^\alpha$ ,  $\lambda_{\alpha\beta}^{g_1, g_2}$  and  $\gamma_\alpha^{g_1, g_2, g_3}$ , and the difference of orbifold actions  $\omega^{g_1, g_2, g_3}$  was classified by  $H^3(\Gamma, U(1))$ .

Firstly we studied the two-form fields  $B$  in the worldvolume gauge theory on the branes. The two-form B-fields are linked to the three-form C-fields, because  $C$  are transformed into  $C + dB$  by the gauge transformations. So we also described the B-fields and their connections on local patches by the two-forms  $B^\alpha$ , the one-forms  $p^{\alpha\beta}$  and the functions  $q_{\alpha\beta\gamma}$ . We wrote down the relation between the C-fields and the B-fields in Eqs.(3.15), (3.16) and (3.17). From these equations and the orbifold group actions (3.5), (3.6), (3.7) and (3.8) on the C-fields we obtained the actions on the B-fields, which are presented as Eqs.(3.18), (3.19) and (3.20), and then we introduced some Čech cochains  $\mu_{\alpha\beta}^g$  and some local one-forms  $\Lambda^{(1)}(g)^\alpha$ . These factors are the structures constructing the orbifold group actions on the B-fields. Calculating the pullbacks for the actions of  $g_1 g_2$  on  $q_{\alpha\beta\gamma}$  and  $p^\alpha$  in two different ways, we derived the constraints (3.26) and (3.27), especially in Eq.(3.27) we added some functions  $\theta_\alpha^{g_1, g_2}$ . In Eq.(3.28) we are able to find that  $\theta_\alpha^{g_1, g_2}$  plays a role similar to discrete torsion in the string theories.

Next we considered the one-form fields  $A$ . We also defined the A-fields on local patches in terms of the one-forms  $A^\alpha$  and the matrices  $g_{\alpha\beta}$ . On the analogy of the string theories we described the relationships between the B-fields and the A-fields as Eqs.(3.21) and (3.22). In order to define the orbifold group actions on  $A^\alpha$  and  $g_{\alpha\beta}$ , we needed some conditions for  $\nu_{\alpha\beta\gamma}^g$  and  $\Lambda^{(1)}(g)^{\alpha\beta}$ , which are the data for the orbifold group actions on the C-fields. The conditions are that  $\nu_{\alpha\beta\gamma}^g$  becomes one and that  $\Lambda^{(1)}(g)^{\alpha\beta}$  vanishes. They are satisfied when the C-fields are topologically trivial, that is, when  $h_{\alpha\beta\gamma\delta} = 1$  and  $v^{\alpha\beta\gamma} = 0$ . Then we obtained the orbifold group actions (3.24) on  $g_{\alpha\beta}$  from Eqs.(3.18) and (3.22), where we introduced some functions  $\rho_\alpha^g$ , and the actions (3.25) on  $A^\alpha$  from Eqs.(3.19), (3.21) and (3.24).

Let us consider the specialized situation so that the C-fields are completely trivial, that is,  $C^\alpha$  is constant,  $u^{\alpha\beta}$  and  $v^{\alpha\beta\gamma}$  vanish and  $h_{\alpha\beta\gamma\delta}$  is equal to one. Since we can take  $\nu_{\alpha\beta\gamma}^g = 1$  from Eq.(3.8), and  $\lambda_{\alpha\beta}^{g_1, g_2} = 1$  from Eq.(3.12), we replace  $\gamma_\alpha^{g_1, g_2, g_3}$  with a globally defined constant  $\gamma^{g_1, g_2, g_3}$ . We also require that the B-fields are completely trivial, in other

words,  $B^\alpha$  becomes globally constant,  $p^{\alpha\beta}$  vanishes and  $q_{\alpha\beta\gamma}$  is equal to one. We can take  $\mu_{\alpha\beta}^g = 1$  from Eq.(3.18) and  $\theta_\alpha^{g_1, g_2}$  are replaced with globally defined  $\theta^{g_1, g_2}$  from Eq.(3.26). And we regard the A-fields as the topologically trivial fields. Since  $g_{\alpha\beta}$  becomes one, from Eq.(3.24) we obtain  $\rho_\alpha^g = \rho_\beta^g$ . After all Eq.(3.28) leads to

$$(\rho^g)(\rho^h) = (\theta^{g,h})(\rho^{gh}), \quad g, h \in \Gamma.$$

This equation implies that the representation of orbifold groups  $\rho^g$  is projective. This result is similar to Eq.(1.2) shown in [3,4].

We were able to have more interesting features in M-theory. Calculating  $\mu_{\alpha\beta}^{g_1, g_2, g_3}$  in two different ways, we obtained Eq.(3.29). In the situation mentioned above we can define  $\theta_\alpha^{g_1, g_2}$  and  $\gamma_\alpha^{g_1, g_2, g_3}$  as global constants, and from Eq.(3.29) we obtain

$$(\theta^{g_1, g_2, g_3})(\theta^{g_1, g_2}) = \gamma^{g_1, g_2, g_3}(\theta^{g_1, g_2, g_3})(\theta^{g_2, g_3}), \quad g_1, g_2, g_3 \in \Gamma.$$

We should recall that  $\omega^{g_1, g_2, g_3}(= \gamma^{g_1, g_2, g_3}/\bar{\gamma}^{g_1, g_2, g_3})$  are classified by  $H^3(\Gamma, U(1))$ . So we assume  $\gamma^{g_1, g_2, g_3}$  as discrete torsion for the branes in M-theory.

In M-theory there exist membranes and M5-branes. Open membranes have end points and end lines on the M5-branes, and in a gauge theory on the worldvolume of  $N$  coincident M5-branes we assume the end lines and the end points as two-form tensor fields and one-form gauge fields respectively. So we considered the two-form B-fields and the one-form A-fields in the three-form C-fields background. When the transverse space for the M5-branes is  $\mathbf{R} \times \mathbf{C}^2$ , the gauge theory on the worldvolume includes two complex scalars and one real scalars as fluctuations of the M5-branes to the transverse directions. When the M5-branes are located at the singularity of orbifold  $\mathbf{R} \times \mathbf{C}^2/\Gamma$ , where  $\mathbf{C}^2/\Gamma$  is ALE space, the gauge theory becomes a six dimensional  $\mathcal{N} = 1$  supersymmetric Yang-Mills theory for large  $N$  [18,19]. Then a tensor multiplet, a hypermultiplet and a vector multiplet consist of the two-form field and the real scalar field, of the complex scalars and of the one-form field respectively. The moduli space of the Yang-Mills theory on the M5-branes at the orbifold singularity with discrete torsion may have good correspondences[20] to the geometric structure of orbifold on the analogy of [7,8].

Though we know that the end points of open strings have Chan-Paton factors in the string theories, the analogues of Chan-Paton factors as the end lines of open membranes are not clear in M-theory. So we do not know precisely what type of values  $\theta^{g_1, g_2}$  and  $\gamma^{g_1, g_2, g_3}$  are. But at least we are able to suggest that the representation  $\rho^g$  of the orbifold

group is projective and that the phase  $\theta^{g_1, g_2}$  has the structure which are determined by the M-theory discrete torsion  $\gamma^{g_1, g_2, g_3}$ .

We will need to make more precise mathematical analyses, for example, quotient stacks [21,22] and K-theory. Dp-branes and M-theory branes are studied in the contexts of K-theory [23] and twisted K-theory [24,25]. Since the p-branes are realized as (p+1)-form fields in low energy effective actions, there may exist some analogues of discrete torsion for the p-branes and the open p-branes.

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